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A Collinear n-Body Problem of Classical Electrodynamics

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Abstract

One model for the motion of n charged particles on the x-axis leads to a system of delay differential equations with delays dependent on the unknown trajectories. If appropriate past histories of the trajectories are given, say on $[\alpha,0]$, then for sufficiently small $t \geq 0$ one has a system of n^2 ordinary differential equations of the form

$$y' = f(t,y) \quad \text{with} \quad y(0) = y_0 \text{ given.} \tag{*}$$

The function f, which involves the known past histories of the trajectories, is continuous; so existence of solutions is assured. However, f does not satisfy the Lipschitz condition usually used for proving uniqueness.

The key new result is that the solution of (*) is unique provided, for some integer $m \leq n^2$,

$$f_i(t,\xi) < 1 \quad \text{for} \quad i = 1,\dots, m, \quad \text{and}$$

$$||f(t,\xi) - f(t,\eta)|| \leq K \sum_{i=1}^{m} |g_i(t-\xi_i) - g_i(t-\eta_i)| + K \sum_{i=m+1}^{n^2} |\xi_i - \eta_i|,$$

where $K > 0$ is constant and each g_i is a continuous function of bounded variation.

This generalized Lipschitz-type condition is indeed satisfied in the electrodynamics case. The m components of y which play the special role in the above uniqueness criterion are the $n(n-1)$ delays of the original n-body problem.

Eventually one finds that solutions of the original equations of motion exist and are unique as long as no two particles collide.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

One model for the motion of n charged particles on the x-axis leads to a system of delay differential equations with delays dependent on the unknown trajectories. If appropriate past histories of the trajectories are given, say on [, 0], then for sufficiently small t 0 one has a system of n^2 ordinary differential equations of the form* $y' = f(t,y)$ with $y(0) = y_0$ given. The function f, which involves the known past histories of the trajectories, is continuous; so existence of solutions is assured. However, f does not satisfy the Lipschitz condition usually used for proving uniqueness. The key new result is that the solution of * is

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A Collinear n-Body Problem of Classical Electrodynamics

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Consider n charged particles on the x-axis; and assume that each is influenced only by the retarded fields of the others, and no two particles collide during the time considered.

Let $x_i(t)$ be the position of particle i at time t, and let c be the speed of light. Then at time t particle j is influenced by the electromagnetic fields which were produced by particle i ($i \neq j$) at some earlier instant $t - r_{ij}(t)$. The delay $r_{ij}(t)$ represents the time required for fields to propagate from particle i to particle j and must satisfy the functional equation

$$cr_{ij}(t) = |x_j(t) - x_i(t - r_{ij}(t))|. \tag{1}$$

To shorten the notation, we will sometimes represent $x_j(t)$ and $r_{ij}(t)$ by x_j and r_{ij}, respectively. Then Eq. (1) becomes

$$cr_{ij} = |x_j - x_i(t - r_{ij})|.$$

We shall also use the symbol

$$v_j = v_j(t) \equiv x_j'(t)/c$$

for the velocity of particle j in units of c. [If t is an endpoint of the interval of definition of x_j, then $x_j'(t)$ is a one-sided derivative.]

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The following theorem, proved in [1], lists some basic properties of Eq. (1).

Theorem 1. Let x_i and x_j be given differentiable functions with $|x_i'(t)| < c$ and $|x_j'(t)| < c$ on $[\alpha,\beta)$, where $\alpha < 0 < \beta$, and with $x_i(t) \neq x_j(t)$ on $[0,\beta)$. Assume Eq. (1) has a solution when $t = 0$. (See Remark below.)

Then Eq. (1) has a unique solution for all $t \in [0,\beta)$ and (1) is equivalent to

$$cr_{ij} = \sigma_{ij}[x_j - x_i(t - r_{ij})], \tag{1'}$$

where

$$\sigma_{ij} \equiv \operatorname{sgn}\,[x_j(0) - x_i(0)].$$

Moreover

$$r_{ij} \geq |x_j - x_i|/2c.$$

Now assume, in addition, that $v_i = x_i'/c$ and $v_j = x_j'/c$ are continuous. Then

$$r_{ij}' = \frac{v_j - v_i(t - r_{ij})}{\sigma_{ij} - v_i(t - r_{ij})} \quad \text{for } 0 \leq t < \beta. \tag{2}$$

Conversely, if r_{ij} is a solution of (2) on $[0,\beta)$ and if $r_{ij}(0)$ satisfies Eq. (1') at $t = 0$, then r_{ij} satisfies (1') on $[0,\beta)$.

Remarks. A sufficient condition to assure that Eq. (1) has a solution when $t = 0$ is

$$|x_i'(t)| \leq c - |x_j(0) - x_i(0)|/|\alpha| \quad \text{for } \alpha \leq t \leq 0.$$

Note from (2) that $d(t - r_{ij})/dt > 0$. So $t - r_{ij}(t)$ is an increasing function of t. In particular, if $\beta < \infty$, then $\lim_{t\to\beta-} r_{ij}(t)$ exists.

After these preliminaries, we obtain the actual equation of motion for particle j as follows. Assume Eq. (1) has a unique solution for each $i \neq j$, sum the retarded fields produced by all other particles $(i \neq j)$, and substitute these fields into the Lorentz-Abraham force law for particle j. The result is

$$\frac{v_j'}{(1 - v_j^2)^{3/2}} = \sum_{i \neq j} \frac{K_{ij}}{r_{ij}^2} \frac{\sigma_{ij} + v_i(t - r_{ij})}{\sigma_{ij} - v_i(t - r_{ij})}, \tag{3}$$

where each K_{ij} is a constant.

The "natural" initial data problem for the system of functional differential equations represented by (1) and (3) is as follows.

Problem P. Let continuously differentiable functions $\phi_1, \ldots, \phi_n$ be given on an interval $[\alpha, 0]$ with the properties that each $|\phi_i'(t)| < c$ on $[\alpha, 0]$ and $\phi_i(0) \neq \phi_j(0)$ when $i \neq j$. A solution of Problem P will be a set of continuously differentiable functions $x_1, \ldots, x_n$ on $[\alpha, \beta)$ for some $\beta > 0$ such that

(a) $x_i(t) = \phi_i(t)$ on $[\alpha, 0]$ for each i,

(b) $x_i(t) \neq x_j(t)$ on $[0, \beta)$ when $i \neq j$,

(c) $|v_i(t)| < 1$ on $[0, \beta)$ for each i, where $v_i \equiv x_i'/c$,

(d) there exists a set of functions $\{r_{ij}\}_{i \neq j}$ on $[0, \beta)$ such that Eqs. (1) and (3) are satisfied on $[0, \beta)$.

As yet, we have made no assumption that the "initial trajectories", $\phi_1, \ldots, \phi_n$, satisfy any particular equations.

In the context of the functional differential equations presented here, it will be quite natural to assume that enough appropriate initial data is given so that each Eq. (1) has a solution at $t = 0$ when $i \neq j$. (This assumption was eliminated in the two-body problem by Travis [6].)

The question is, do we need further smoothness conditions on ϕ_i in order to establish the existence and uniqueness of a solution of Problem P?

When this problem was originally treated for the case of two particles, it was assumed that each ϕ_i' was Lipschitz continuous [1]. But later consideration of the problem in three dimensions --involving equations of neutral type--suggested that this was an unreasonable hypothesis [2]. It seemed more realistic to require each ϕ_i' to be just absolutely continuous.

For the collinear motion of two particles, mere continuity of each ϕ_i' turned out to be adequate [4]. Then, for the case of two particles moving in three dimensions it was found that absolute continuity is sufficient for local existence and uniqueness [3].

However, the proofs used in [4] and [3] do not extend to the case $n \geq 3$.

To resolve this difficulty the following uniqueness theorem for ordinary differential equations has recently been proved [5].

Theorem 2. The system of p equations

$$y' = f(t,y) \quad \text{with} \quad y(0) = y_0$$

has a unique solution (locally) if, in some neighborhood of $(0,y_0)$ in R^{1+p}, f is continuous and for some integer $m \in [0,p]$

$$f_i(t,\xi) < 1 \quad \text{for} \quad i = 1, \ldots, m$$

and f satisfies the generalized Lipschitz-type condition

$$||f(t,\xi) - f(t,\eta)|| \leq K \sum_{i=1}^{m} |g_i(t-\xi_i) - g_i(t-\eta_i)| + K \sum_{i=m+1}^{p} |\xi_i - \eta_i|,$$

where $||\cdot||$ is any norm in R^p, $K > 0$ is a constant, and each g_i: $R \to R$ is continuous and is of bounded variation on bounded subintervals.

The theorem proved in [5] is actually more general, but the special case above will suffice for our present purpose.

The following (main) theorem says that the collinear n-body problem does have a unique solution (until a collision occurs) even assuming only absolutely continuous initial velocities--or slightly less.

Theorem 3. Let $\phi_1, \ldots, \phi_n$ be given functions on $[\alpha, 0]$ with each ϕ_i' continuous and of bounded variation. Assume that

(i) $\phi_j(0) \neq \phi_i(0)$ when $j \neq i$,

(ii) $|\phi_i'(t)| < c$ on $[\alpha, 0]$ for each i, and

(iii) Eq. (1) has a solution $r^*_{ij}(0)$ at $t = 0$ for each i and $j \neq i$.

Then there exists $\beta > 0$ such that Problem P has a unique solution on $[\alpha, \beta)$, and either $\beta = +\infty$ or else for some i and $j \neq i$

$$\lim_{t \to \beta-} x_j(t) = \lim_{t \to \beta-} x_i(t) \quad \text{-- a collision.}$$

Proof. For economy of notation, let r represent the $n(n-1)$-vector-valued function with components r_{ij}.

Note that the last paragraph of Theorem 1 essentially lets us consider Eqs. (2) and (3) in place of (1) and (3). The solutions of Eqs. (2) and (3) which will be of interest must satisfy $(t, r, v_1, \ldots, v_n) \in D$, where

$$D \equiv R \times (0, \infty)^{n(n-1)} \times (-1, 1)^n \subset R^{1+n^2}.$$

At first consider the open subset

$$U \equiv \{(t, \xi) \in D: \ t - \xi_i < 0 \ \text{ for } \ i = 1, \ldots, n(n-1)\}.$$

Note that $(0, r^*(0), \phi_1'(0)/c, \ldots, \phi_n'(0)/c) \in U$.

Now define $g_{ij} = \phi_i'/c$ for each i and $j \neq i$. Then in U Eqs. (2) and (3) are equivalent to the ordinary differential equations

$$r_{ij}' = \frac{v_j - g_{ij}(t - r_{ij})}{\sigma_{ij} - g_{ij}(t - r_{ij})} \tag{4}$$

and

$$\frac{v_j'}{(1 - v_j^2)^{3/2}} = \sum_{i \neq j} \frac{K_{ij}}{r_{ij}^2} \frac{\sigma_{ij} + g_{ij}(t - r_{ij})}{\sigma_{ij} - g_{ij}(t - r_{ij})}, \tag{5}$$

and the initial conditions are $r_{ij}(0) = r_{ij}^*(0)$, $v_j(0) = \phi_j'(0)/c$. Local existence and uniqueness of a solution, say for $0 \leq t < h$, follow from Theorem 2 with $p = n^2$ and $m = n(n-1)$. We have thus generated the unique solution of Eqs. (2) and (3) on $[0, h)$. Now let

$$x_i(t) = \begin{cases} \phi_i(t) & \text{on } [\alpha, 0] \\ \phi_i(0) + \int_0^t cv_i(s)\, ds & \text{on } [0, h). \end{cases}$$

Then the resulting functions $x_1, \ldots, x_n$ form the unique solution of Problem P on $[\alpha,h)$.

Next define

$$\beta = \sup \{s \geq 0\text{: a unique solution exists on } [\alpha,s)\}.$$

Then $\beta > 0$ and a unique solution, $x_1, \ldots, x_n$, of Problem P can be constructed on $[\alpha,\beta)$. Specifically, for each $t \in (0,\beta)$, define $x_i(t) = y_i(t)$ where $y_1, \ldots, y_n$ is any solution valid on $[\alpha,s)$ for some $s > t$.

Suppose (for contradiction) that $\beta < \infty$ and there exists $\delta > 0$ such that

$$|x_i(t) - x_j(t)| \geq \delta \text{ on } [0,\beta) \text{ for each } i \text{ and } j \neq i. \tag{6}$$

It follows from Theorem 1 that each

$$r_{ij}(t) \geq \delta/2c.$$

Hence there exists $a \in [0,1)$ such that

$$|v_i(t - r_{ij})| \leq a \quad \text{on} \quad [0,\beta) \quad \text{for each} \quad i \quad \text{and} \quad j \neq i.$$

Using these results, we find from Eq. (3) that

$$\frac{d}{dt} \frac{v_j}{(1 - v_j^2)^{1/2}} = \frac{v_j'}{(1 - v_j^2)^{3/2}} \quad \text{is bounded on} \quad [0,\beta) \quad \text{for each} \quad j.$$

Since $\beta < \infty$, this means that for some $b \in [0,1)$

$$|v_j(t)| \leq b \quad \text{on} \quad [0,\beta) \quad \text{for each} \quad j.$$

Now, defining

$$x_i(\beta) = \lim_{t \to \beta-} x_i(t) \quad \text{and} \quad r^*_{ij}(\beta) = \lim_{t \to \beta-} r_{ij}(t),$$

we obtain functions $x_1, \ldots, x_n$ having the same properties on $[\alpha,\beta]$ as were originally assumed for $\phi_1, \ldots, \phi_n$ on $[\alpha,0]$. Here $r^*_{ij}(\beta)$ plays the role of $r^*_{ij}(0)$. This means one can construct a unique extension of the solution beyond $t = \beta$ (at least for a short interval) just as we did originally beyond $t = 0$. And that contradicts the definition of β.

So if $\beta < \infty$, it follows that (6) fails and a collision occurs at β.

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